# $d$ dimensional $SO(d)$-Higgs models with an instanton and sphaleron: $d=2,3$

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The Abelian Higgs model and the Georgi-Glashow model in two and three Euclidean dimensions, respectively, support both finite-size instantons and sphalerons. The instantons are the familiar Nielsen-Oleson vortices and the 't Hooft-Polyakov monopole solutions, respectively. We have constructed the sphaleron solutions and calculated the Chern-Simons charges $\mathcal{N}_{CS}$ for sphalerons of both models and have constructed two types of noncontractible loops between topologically distinct vacua. In the three-dimensional model, the sphaleron and the vacua have a zero magnetic and electric flux while the configurations on the loops have a nonvanishing magnetic flux. [S0556-2821(99)02922-7]

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## I. INTRODUCTION

In the semiclassical treatment of quantum field theories, instantons [1–3] play the important role of providing the tunneling between topologically inequivalent vacua in an essentially nonperturbative framework. Instantons are classical solutions to the Euler-Lagrange equations with Euclidean time, from which comes the tunneling interpretation. Another, closely related mechanism for vacuum to vacuum transitions but now at nonzero temperature is provided by the (Euclidean) static solutions of the field equations called sphalerons [4]. In contrast with instantons, sphalerons are unstable solutions and provide a classical rather than quantum (tunneling) transition over the energy barrier separating the two vaccua. In this context it is very useful to consider another class of classical solutions, which are periodic in (Euclidean) time, with the period being identified with the inverse of the temperature. These are the periodic instantons as defined in Ref. [6]. Thus at zero temperature the period of the periodic instanton becomes infinite, rendering it nonperiodic, which can be identified as the instanton itself.

Now a given field theoretic model supporting sphalerons [4,5] and hence also periodic instantons [6] may or may not support an (zero-temperature or infinite period) instanton. In the case where a finite-size instanton exists, it can be arrived at as the period of the periodic instanton tends to infinity. In the case, however, where no finite-size instanton exists, the situation is more complicated and the so-called constrained instantons [7] must be employed. The consequences of a theory being of one type or the other are thought to be potentially important.

There has been some exploratory work done in this direction, in the context of the $(1+1)$-dimensional scale-breaking $O(3)$ sigma model [8] which does not support finite-size instantons and its Skyrmed version [9] which does so. The study of the periodic instantons in these two models from this viewpoint was carried out in Refs. [10,11], respectively.

In view of the above, it is interesting to construct and study models that can support both *instantons* and *sphalerons*, as a first step before studying the *periodic instantons* interpolating them. Our aim in this paper is to do just this, for a class of $d$-dimensional $SO(d)$-Higgs models in which the Higgs field is a $d$-component vector of the $SO(d)$, for the two cases of $d=2$ and $d=3$. What distinguishes such models is that their instantons result in curvature field strengths that exhibit *inverse square* behavior. This property contrasts with the pure-gauge behavior of the arbitrary-scale Yang-Mills (YM) instantons [1] and can result in far-reaching (physical) consequences. In the $d=3$ case, it leads to a dilute Coulomb gas of instantons, as shown by Polyakov [3] long ago, while the corresponding instantons in $d=4$ share this property [12] and can possibly also enable the construction of a dilute Coulomb gas [13]. This is our physical justification for making a systematic study of these $SO(d)$-Higgs models, and we concentrate on the $d=2,3$ cases here. Thus both models under consideration here are the familiar ones, namely, the Abelian Higgs model in $d=2$ and the Georgi-Glashow model in $d=3$.

The instantons in these two $(1+1)$- and $(2+1)$-dimensional models are the well-known topologically stable Nielsen-Olesen vortices [14] and the 't Hooft-Polyakov monopoles [15], respectively. It is therefore the sphaleron solutions to these models that are the remaining

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entities to be studied. In this connection, it is true that the sphalerons of the Abelian Higgs model were studied extensively long ago [16,17], but we repeat it here for the sake of completeness so that both cases $d=2$ and $d=3$ are treated similarly and because the presentation of the results in the literature can be refined. In particular we clarify the situation with respect to the question of periodic boundary conditions in the spacelike coordinate used in the literature [16,17]. The sphaleron in the $d=3$ case has been studied recently [18]. We have carried out the analysis of the sphaleron solutions in both models employing both the noncontractible loop (NCL) used by Manton [4] for the Weinberg-Salam model, to which we refer in this paper as the *geometric* loop construction, as well as the finite-energy path method of Akiba *et al.* [25], to which we refer as *boundary* loop construction.

In the study of the sphaleron solutions, a central role is played by the Chern-Simons number, which in the Weinberg-Salam model is calculated from the second Chern-Pontryagin (CP) density. In the $d$-dimensional Higgs models we consider, the natural candidate for the latter is the dimensionally reducing CP density of the YM field on $\mathbb{R}^d \times S^{4p-d}$, where $4p>d$ [19]. The $d$-dimensional model is decided by $p$ such that the action density is bounded from below by dimensionally reducing the CP density in question. The simplest such model corresponds to the case where $4p$ is the smallest number greater than $d$. Both the Abelian Higgs model and the Georgi-Glashow models considered in this work are the simplest models whose CP densities are arrived at by the corresponding dimensional descent of the *second* CP density of the $SU(2)$ YM field.

The analysis of the Abelian Higgs system in two dimensions is presented in Sec. II. In Secs. II A and II C, respectively, the *geometric* and *boundary* loop constructions are presented, while in Sec. II B a discussion of our constructions is contrasted with what is usually given in the literature, and the differences are commented on.

The analysis of the Georgi-Glashow model in three dimensions is presented in Sec. III. In Sec. III A the *boundary* loop construction is presented. The *geometric* loop construction is presented in Sec. III B. Since the sphaleron solution of this model turns out to be effectively the solution of an Abelian Higgs model, a family of arbitrary vorticity $N$ sphalerons are constructed in Sec. III B. Section III C is devoted to a brief discussion of the magnetic and electric properties of this sphaleron and the configurations on the geometric loop. In Sec. IV, we give a summary of our results.

## II. $d=2$: SPHALERONS IN THE ABELIAN HIGGS MODEL

Our objective in this section is to construct noncontractible loops between two topologically neighboring vacua which feature the sphaleron at the top of the energy barrier. Our constructions run exactly parallel to those of Manton [4] and Akiba *et al.* [25], respectively. This differs from the analysis of Bochkarev and Shaposhnikov [16] in that the latter employ periodicity in the space variable to construct the NCL.

The $SO(2)$ Abelian Higgs model in $d=2$ spacetime dimensions is given by the Euclidean Lagrange density

$$\mathcal{L}=\frac{1}{4}F_{\mu\nu}^2+\frac{1}{2}|D_\mu\varphi|^2+\frac{\lambda_0}{32}(1-|\varphi|^2)^2, \tag{1}$$

with $\varphi=\phi^1+i\phi^2$, $D_\mu\varphi=\partial_\mu\varphi+iA_\mu\varphi$; i.e., we used the identity $SO(2)=U(1)$ to write the Higgs doublet $\phi^i$ as one complex field.

Adding the inequalities

$$\left(F_{\mu\nu}-\frac{\sqrt{\lambda_0}}{4}\epsilon_{\mu\nu}(1-|\varphi|^2)\right)^2\geqslant 0,\quad |D_\mu\varphi-i\epsilon_{\mu\nu}D_\nu\varphi|^2\geqslant 0, \tag{2}$$

the cross terms of the squares yield a lower bound for the Euclidean action, $\mathcal{L}\geqslant\varrho$, which becomes a topological lower bound; i.e., $\varrho$ can be written as a total divergence,

$$\varrho=\partial_\mu\Omega_\mu,\quad \Omega_\mu=\frac{1}{4}\epsilon_{\mu\nu}(A_\nu+i\varphi^*D_\mu\varphi), \tag{3}$$

if $\lambda_0=1$. $\Omega_\mu$ is the Chern-Simons form of this model. It consists of a gauge-dependent and a gauge-independent part which is a typical feature of $SO(d)$-Higgs theories in even spacetime dimensions [19].

The topological lower bound (which can be generalized to $\lambda_0\neq 1$) ensures the stability of the instantons of the model, the Abelian Higgs vortex, which can be found using the radially symmetric ansatz

$$\varphi=h(r)e^{-iN\theta},\quad A_\mu=\frac{a(r)-N}{r}\epsilon_{\mu\nu}\hat{x}_\nu, \tag{4}$$

with $r^2=x_0^2+x_1^2=t^2+x^2$, and integrating the Euler-Lagrange equations of the resulting radial subsystem Lagrangian:

$$L_0=\pi\left\{\frac{1}{r}a'^2+r\left(h'^2+\frac{a^2h^2}{r^2}\right)+\frac{\lambda_0}{16}(1-h^2)^2\right\}. \tag{5}$$

The corresponding instanton solutions are self-dual provided $\lambda_0=1$; then they saturate the inequality $\mathcal{L}\geqslant\partial_\mu\Omega_\mu$ and hence are characterized by integer Chern-Simons number (also called Chern-Pontryagin charge in the context of instanton physics)

$$\mathcal{N}_{CS}=\frac{2}{\pi}\int_{-\infty}^{\infty}dt\int_{-\infty}^{\infty}dx\,\partial_\mu\Omega_\mu=\frac{2}{\pi}\int_{\mathbb{R}^2}\partial_\mu\Omega_\mu d^2x$$
$$=\frac{2}{\pi}\lim_{r\to\infty}\int_{S^1}\Omega_\mu dS_\mu. \tag{6}$$

This can be arrived at by subjecting the second Chern-Pontryagin class of the $SU(2)$ YM field on $\mathbb{R}^2\times S^2$ to dimensional descent [19].

The Chern-Simons number depends only on the behavior of the instanton at infinity, reflecting the topological properties of the mapping:

$$\varphi^{\infty}_{inst}:S^1_{spacetime}\rightarrow S^1_{Higgs}. \tag{7}$$

In contrast with the instanton, the sphaleron is a ‘‘*static*’’ object; i.e., it does not depend on the Euclidean time. Nonetheless, it is an *Euclidean* spacetime object closely related to the instanton as it is the extremum of the energy functional of a set of static configurations connecting distinct vacua in a topologically nontrivial way, i.e., a *noncontractible loop*. The NCL, parametrized in terms of Euclidean time, is an instantonlike object and the loop as a whole has integer topological number. The Chern-Simons number $\mathcal{N}_{CS}$ is also parametrized in terms of the Euclidean time,

$$\mathcal{N}_{CS}(t_0)=\frac{2}{\pi}\int_{-\infty}^{t_0}dt\int_{-\infty}^{\infty}dx\,\partial_\mu\Omega_\mu; \tag{8}$$

hence the loop can be parametrized in terms of $\mathcal{N}_{CS}$, the latter taking on all values between 0 and 1.

To construct and study the sphaleron, we start from the static energy which in the temporal gauge ($A_0=0$, $A:=A_1$) is given by

$$\mathcal{E}[\varphi,A]=\int\left[\frac{1}{2}|(\partial_x+iA)\varphi|^2+\frac{1}{32}(1-|\varphi|^2)^2\right]dx. \tag{9}$$

Choosing the particular ansatz $\varphi=\phi\in\mathbb{R}$, $A=0$, this reduces to the well-known real $\varphi^4$ energy functional

$$\mathcal{E}_{sph}[\phi]=\int\left[\frac{1}{2}(\phi')^2+\frac{1}{32}(1-\phi^2)^2\right]dx, \tag{10}$$

which is minimized by the kink-antikink solutions

$$\phi_\pm(x)=\pm\tanh\left(\frac{x-x_0}{4}\right). \tag{11}$$

The $\phi^4$ kink on its own is a stable soliton, but it becomes unstable as soon as it is embedded in complex isospace which can be shown explicitly by investigating the fluctuation spectrum [20].

### A. The geometrical loop construction

To show that $(\varphi,A)_{sph}=(\phi_\pm,0)$ are sphaleron configurations of the Abelian Higgs model, we construct the corresponding NCLs of finite energy configurations connecting two vacua through the sphaleron in a topologically nontrivial way.

*Vacua* are static configurations with zero energy; i.e., they are given by $\varphi=g$, $A=ig^{-1}\partial_x g$ with $g\in U(1)$. We use the remaining gauge freedom of the theory to fix the vacuum to $(\varphi,A)_{vac}=(1,0)$ and comment on the choice of gauge later on.

Following the geometrical loop construction of Manton [4], we consider first the topological properties of the Higgs field at infinity. One-dimensional space at infinity shrinks to the discrete set $(\mathbb{R})^\infty=S^0_{space}=\{\pm1\}$. Hence one has to distinguish two mappings $\varphi^{+\infty}$, $\varphi^{-\infty}$ instead of one mapping $\varphi^\infty$ which depends on continuous angular coordinates in higher space dimensions. To construct a geometrical NCL [4], we introduce a single loop parameter $\tau\in S^1_{loop}$ as an additional degree of freedom such that

$$\varphi^{+\infty}:S^1_{loop}\rightarrow S^1_{Higgs} \tag{12}$$

is a topologically nontrivial mapping which we choose to be

$$\varphi^{+\infty}:\tau\mapsto e^{2i\tau}, \tag{13}$$

whereas $\varphi^{-\infty}\equiv1$ is chosen to be topologically trivial. The finite energy condition then also fixes the gauge field at infinity as the covariant space derivative in Eq. (9) has to vanish; hence,

$$A^{\pm\infty}:=A(r\rightarrow\pm\infty)=-i(\varphi^{\pm\infty})^{-1}\partial_x\varphi^{\pm\infty}=0. \tag{14}$$

The general Manton loop ansatz constructed using the topological ingredients $\varphi^{+\infty}$, $A^{\pm\infty}$ is now given by

$$\overline{\varphi}=[1-h(x)]\psi+h(x)\varphi^{+\infty},\quad \overline{A}=fA^{\pm\infty}=0, \tag{15}$$

where $\psi$ has to be chosen such that the loop starts and ends in the vacuum and reaches the sphaleron for $\tau=\pi/2$; hence $\overline{\varphi}|_{\tau=0}=\overline{\varphi}|_{\tau=\pi}=1$, $\overline{\varphi}|_{\tau=\pi/2}=h\varphi^{+\infty}|_{\tau=\pi/2}=-h$. In the $SO(2)$ model,

$$\psi=\cos^2\tau+i\sin\tau\cos\tau \tag{16}$$

is a convenient choice. Moreover, $\overline{\varphi}(x\rightarrow\pm\infty)=\varphi^{\pm\infty}$ requires $h(x\rightarrow\pm\infty)=\pm1$. The loop is then given by

$$\overline{\varphi}(\tau,x)=e^{i\tau}[\cos\tau+ih(x)\sin\tau],\quad \overline{A}\equiv0. \tag{17}$$

Inserting the ansatz (17) into the static energy functional (9) yields

$$\mathcal{E}[\overline{\varphi},\overline{A}]=\overline{\mathcal{E}_\tau}[h]=\sin^2\tau\int\left[\frac{1}{2}(h')^2+\frac{1}{32}\sin^2\tau(1-h^2)^2\right]dx, \tag{18}$$

which for $\tau=\pi/2$ reduces to the $\varphi^4$ model, $\overline{\mathcal{E}_{\tau=\pi/2}}=\mathcal{E}_{sphal}=\frac{1}{3}$.

For any fixed value of $\tau\in[0,\pi]$, $\overline{\mathcal{E}_\tau}[h]$ is *minimized* by

$$h_\tau=\tanh\left(\sin\tau\frac{x-x_0}{4}\right). \tag{19}$$

The energy along the resulting minimal energy loop is

$$\overline{\mathcal{E}}(\tau)=\frac{1}{3}\sin^3\tau, \tag{20}$$

which has a *maximum* for $\pi/2$. This minimax procedure therefore shows that $(\varphi,A)_{sph}=(\phi_-,0)$ is a sphaleron of the Abelian Higgs model. To construct the NCL for the kink-type sphaleron $(\varphi,A)_{sph}=(\phi_+,0)$, one simply has to use

$$\varphi^{-\infty}:\tau\mapsto e^{2i\tau} \tag{21}$$

as the topologically nontrivial mapping.

To calculate the increase of the Chern-Simons number along the NCL, one has to treat the loop parameter as a (Euclidean) time-dependent quantity $\tau=\tau(t)$ with $\tau(t=-\infty)=0$, $\tau(t=\infty)=\pi$. Inserting the loop ansatz (17) into Eq. (8), one can split the double integral into a space ''volume'' and a ''surface'' integral, resulting in

$$\mathcal{N}_{CS}(t_0)=\frac{2}{\pi}\left\{\int_{-\infty}^{\infty}\Omega_0\Big|_{t=-\infty}^{t=t_0}dx+\int_0^{t_0}\Omega_1\Big|_{x=-\infty}^{x=+\infty}dt\right\}$$
$$=\frac{\tau(t_0)}{\pi}+\frac{1}{2\pi}\sin 2\tau(t_0). \qquad (22)$$

As expected, one finds $\mathcal{N}_{CS}=\frac{1}{2}$ when the loop reaches the sphaleron, whereas the entire loop has $\mathcal{N}_{CS}=1$.

### B. Gauging and nonperiodic boundary conditions

The vacuum convention chosen above can be changed by *gauging the geometrical loop,*

$$\overline{\varphi}\mapsto\check{\varphi}=g\overline{\varphi}, \quad \overline{A}=0\mapsto\check{A}=ig^*\partial_x g. \qquad (23)$$

One could, e.g., require $\check{\varphi}=g\overline{\varphi}\to 1$ for $x\to-\infty$ *and* $x\to+\infty$, choosing the gauge [20]

$$g=e^{-i\tau\Lambda(x)}, \quad \Lambda(x)=\frac{2}{\pi}\arctan\left(\frac{x}{\alpha}\right)+1 \qquad (24)$$

($\alpha\neq 0$ arbitrary), which yields

$$A(x)=\frac{2\tau}{\pi}\frac{\alpha}{\alpha^2+x^2}.$$

Concerning its $\tau$ dependence, $g$ can be considered either as a set of static transformations parametrized by $\tau$ and applied to each corresponding static configuration along the loop separately or as one time-dependent gauge transformation applied to the $\tau(t)$ time-dependent loop as a whole. In the latter case, the temporal gauge condition is violated, $A_0\equiv 0\mapsto\dot{\tau}\Lambda$.

In both cases, one finds for the gauge-transformed Chern-Simons form $\Omega_0\mapsto\Omega_0+\frac{1}{4}\tau\Lambda'$, $\Omega_1\mapsto\Omega_1-\frac{1}{4}\dot{\tau}\Lambda$. Inserting this into Eq. (22), we find explicitly that the Chern-Simons number is gauge invariant; i.e., also the gauged Manton loop $(\check{\varphi},\check{A})$ starts at a vacuum with $\mathcal{N}_{CS}(\tau=0)=0$, reaches the sphaleron at $\mathcal{N}_{CS}(\tau=\pi/2)=\frac{1}{2}$, and ends at a vacuum with $\mathcal{N}_{CS}(\tau=\pi)=1$. This is not surprising, as $\varrho=\partial_\mu\Omega_\mu$ is a gauge-invariant quantity.

The main new feature of the gauged geometrical loop is the representation of the vacuum states between which the loop interpolates. Generalizing the gauged geometrical loop $(\check{\varphi},\check{A})$ to a larger range of the loop parameter $\tau\in\mathbb{R}$, the loop reaches a vacuum state for any $\tau=n\pi$, $n\in\mathbb{Z}$, now given by

$$\check{\varphi}^{(n)}=\left(-\frac{\alpha-ix}{\alpha+ix}\right)^n, \quad \check{A}^{(n)}=\frac{n\alpha}{\alpha^2+x^2}. \qquad (25)$$

These vacua can be labeled by a ''topological charge'' defined as

$$Q:=\frac{2}{\pi}\int\Omega_0 dx \;\Rightarrow\; Q|_{(\check{\varphi}^{(n)},\check{A}^{(n)})}=\frac{1}{2\pi}\int\check{A}^{(n)}dx=n. \qquad (26)$$

In the literature [16,17] this is usually the only ''topological number'' discussed in the context of the $SO(2)$ Abelian Higgs sphaleron. It should be stressed that it is different from the Chern-Simons number which is the proper quantity to be investigated in the context of the sphaleron NCL and its relation to the instanton. The topological charge used above to label the vacua makes use of the gauge-variant part of the Chern-Simons form. Only in even dimensions does the Chern-Simons form exhibit a gauge-variant part [19]. In these dimensions one can distinguish between ''small'' and ''large'' gauge transformations, depending on whether they leave the ''topological charge'' (defined as the space volume integral of the zero component of the Chern-Simons form) invariant or not.

Another confusion about the $SO(2)$ Abelian Higgs model found in the literature [16,17] concerns the use of periodic boundary conditions in the space coordinates for the gauge fields, i.e., ''putting the fields on the circle'' instead of using the nonperiodic vacuum structure (25) [20]. The sphaleron, however, is definitely never a periodic object in space, but always the $\varphi^4$ kink, a contradiction which cannot be solved by simpy taking the limit of infinite period.

The periodic solutions of $\varphi^4$ theory [21] and Goldstone theory [22] in one dimension can be more gainfully interpreted as the *periodic instantons in Euclidean quantum mechanics*. These periodic solutions describe tunneling from thermally excited states, the temperature being given by the inverse period. At some value of the period, the periodic instantons reduce to the (time-independent) sphaleron which in the case of Euclidean quantum mechanics is just a constant solution. This effect describes the phase transition between classical and quantum behavior which is presently under intense investigation [23]. Since periodic instantons (periodic in the Euclidean time, not in the spatial coordinate) are also known to exist in the Abelian Higgs model [24], one should be even more careful about the notion of periodicity in this model.

### C. AKY boundary loop construction

A different technique for the construction of a NCL for the Abelian Higgs sphaleron is motivated by the ''*static minimal energy path*'' construction by Akiba, Kikuchi, and Yanagida (AKY) [25] in the Weinberg-Salam theory. This NCL is constructed by minimizing a general spherically symmetric static ansatz with parameter-dependent boundary conditions, i.e., not the loop ansatz containing the loop parameter, but the boundary conditions. We shall refer to this type of construction as a ''boundary loop.''

The spherically symmetric AKY ansatz in one space dimension (in temporal gauge) is simple and only puts some restriction on the symmetry properties of the parameter functions,

$$\tilde{\varphi}(x)=H(|x|)+i\hat{x}K(|x|), \quad \tilde{A}(x)=f(x), \tag{27}$$

with $\hat{x}=\mathrm{sgn}(x)$; i.e., $H(x)=H(|x|)$ is an even and $K(x)=\hat{x}K(|x|)$ an odd function of $x\in\mathbb{R}$. Under a gauge transformation $g=e^{i\Lambda(x)}$, $H$ and $K$ are rotated,

$$H\mapsto H\cos\Lambda-K\sin\Lambda, \quad K\mapsto K\cos\Lambda+H\sin\Lambda, \tag{28}$$

while $f\mapsto f-\Lambda'$; i.e., one can gauge $f=0$ without loss of generality in the ansatz (27).

Inserting the ansatz into the static energy functional yields

$$\mathcal{E}[\tilde{\varphi},\tilde{A}]=\tilde{\mathcal{E}}[H,K] = \int\left[\frac{1}{2}(H'+K')^2+\frac{1}{32}(1-H^2-K^2)^2\right]dx; \tag{29}$$

i.e., finite energy requires

$$H(x\to\infty)=\cos q, \quad K(x\to\infty)=\sin q. \tag{30}$$

Extrema of $\tilde{\mathcal{E}}[H,K]$ with these boundary conditions (and $H$ even, $K$ odd) are found only for $q=0$ or $q=\pi$, yielding the vacua $H=\pm 1$, $K=0$, and for $q=\pi/2$ with $H=0$, $K(x)=\phi_+(x)=\tanh(x/4)$ which is the sphaleron configuration.

The idea now is to construct a loop by increasing the boundary condition parameter $q(t)$ as a time-dependent quantity from $q(t=-\infty)=0$ to $q(t=\infty)=\pi$. Inserting the ansatz in the Chern-Simons functional (22) and taking care of the boundary conditions for $H$, $K$ when evaluating $\Omega_1|_{x=-\infty}^{x=+\infty}$ yields

$$\mathcal{N}_{CS}=\frac{1}{2\pi}\int(KH'-HK')dx+\frac{q}{\pi}; \tag{31}$$

i.e., the sphaleron has $\mathcal{N}_{CS}=\frac{1}{2}$. One can minimize the static energy functional (29) for fixed Chern-Simons number by adding the Chern-Simons functional with a Lagrange multiplier $\xi$ to the static energy functional, leading to the Euler-Lagrange equations

$$H''+\xi K'+\frac{1}{8}H(1-H^2-K^2)=0,$$

$$K''-\xi H'+\frac{1}{8}H(1-H^2-K^2)=0, \tag{32}$$

which are solved with the above boundary conditions by

$$H(x)=\cos q, \quad K(x)=\sin q\tanh\left(\frac{\sin q}{4}x\right), \quad \xi=8\cos q. \tag{33}$$

The energy along this loop is found to be

$$\tilde{\mathcal{E}}(q)=\frac{1}{3}\sin^3 q, \tag{34}$$

and the Chern-Simons number along the AKY loop for the Abelian Higgs model is

$$\tilde{\mathcal{N}}_{cs}(q)=\frac{q}{\pi}+\frac{1}{2\pi}\sin 2q. \tag{35}$$

Therefore, the energy along the loop, $\tilde{\mathcal{E}}(\tilde{\mathcal{N}}_{cs})$, agrees with the result (20),(22) for the geometrical NCL. Hence there is no advantage of a lower energy along the boundary loop compared to the geometrical loop; both provide equivalent ‘‘static minimal energy paths’’ [25].

## III. $d=3$: SPHALERONS IN THE GEORGI-GLASHOW MODEL

The model is described by the $SO(3)$ taking its values in the $SU(2)$ algebra with anti-Hermitian generators $(-i/2)(\sigma_\mu)=(-i/2)\vec{\sigma}$ and a Higgs triplet field $(\phi^\mu)=\vec{\phi}$ which we write in anti-Hermitian isovector representation, $\Phi=\vec{\phi}\cdot[(-i/2)\vec{\sigma}]$, in $d=3$ spacetime dimensions. The model is given by the Euclidean Lagrangian

$$\mathcal{L}=\mathrm{tr}\left[-\frac{1}{4}F_{\mu\nu}^2-\frac{1}{2}(D_\mu\Phi)^2+\frac{\lambda}{2}\left(\Phi^2+\frac{\eta^2}{2}\right)^2\right] \tag{36}$$

$$=\frac{1}{2}\left[\frac{1}{4}\vec{F}_{\mu\nu}^2+\frac{1}{2}(D_\mu\vec{\phi})^2+\frac{\lambda}{8}(\vec{\phi}^2-2\eta^2)^2\right], \tag{37}$$

with $F_{\mu\nu}=\partial_{[\mu}A_{\nu]}+[A_\mu,A_\nu]=\vec{F}_{\mu\nu}\cdot[(-i/2)\vec{\sigma}]$ and $D_\mu\Phi=\partial_\mu\Phi+[A_\mu,\Phi]\Rightarrow D_\mu\vec{\phi}=\partial_\mu\vec{\phi}+\vec{\phi}\wedge\vec{A}_\mu$, with $\mu=1,2,3$.

From the second form of the Lagrangian [3] it is easy to identify the particle spectrum of the theory which consists of photons, heavy charged bosons with mass $m_W=\sqrt{2}\eta$, and scalar neutral Higgs particles with mass $m_H=\sqrt{2\lambda}\,\eta$.

The model was previously exploited by Polyakov [3] to calculate quark confinement effects, using a dilute gas of instantons. The instanton of the model is the finite size spherically symmetric ’t Hooft–Polyakov monopole [15] in three dimensions which is characterized by integer Chern-Simons number

$$\mathcal{N}_{CS}=\frac{1}{\sqrt{2}\pi\eta}\int\partial_\mu\Omega_\mu d^3x \quad \text{with} \quad \Omega_\rho=\frac{1}{4}\epsilon_{\rho\mu\nu}\,\mathrm{tr}[\Phi F_{\mu\nu}]. \tag{38}$$

As the model is in an odd spacetime dimension, the Chern-Simons form $\Omega_\mu$ in this model, which is constructed from the Bogomol’nyi inequality

$$\mathrm{tr}[F_{\mu\nu}-\epsilon_{\mu\nu\rho}D_\rho\Phi]^2>0\Rightarrow\mathcal{L}>\partial_\mu\Omega_\mu, \tag{39}$$

has no gauge-variant part [19].

To discuss the sphalerons of the model [18], we reduce the Lagrangian (36) to the static Hamiltonian

$$\mathcal{H}=\mathrm{tr}\left[-\frac{1}{4}F_{ij}^2-\frac{1}{2}(D_i\Phi)^2+\frac{\lambda}{2}\left(\Phi^2+\frac{\eta^2}{2}\right)^2\right], \tag{40}$$

with $i,j,\ldots=1,2$.

### A. Sphaleron and boundary loop construction

Motivated by the AKY technique in the Weinberg-Salam model [25], we try the general static spherically symmetric ansatz for the Higgs field to find the sphaleron and a corresponding NCL simultaneously,

$$\Phi = i\frac{\eta}{\sqrt{2}}[H(r)\sigma_3 + K(r)\hat{x}_i\sigma_i], \tag{41}$$

with $r^2 = x_1^2 + x_2^2$.

The $SO(2)$ gauge transformation $\Phi \mapsto g_\Lambda{}^{-1}\Phi g_\Lambda$ with

$$g_\Lambda = \exp\left\{\frac{i}{2}\epsilon_{ij}\hat{x}_i\sigma_j\Lambda(r)\right\} \tag{42}$$

rotates the parameter functions,

$$H \mapsto H\cos\Lambda - K\sin\Lambda, \quad K \mapsto K\cos\Lambda + H\sin\Lambda. \tag{43}$$

This motivates the choice of a spatial spherically symmetric ansatz

$$A_i = \frac{f_A + 1}{r}\epsilon_{ik}\hat{x}_k\left(-\frac{i}{2}\sigma_3\right) + \frac{f_B}{r}\epsilon_{ij}\hat{x}_j\left(-\frac{i}{2}\hat{x}_k\sigma_k\right) + \frac{f_C}{r}\hat{x}_i\left(-\frac{i}{2}\epsilon_{kl}\hat{x}_k\sigma_l\right) \tag{44}$$

for the gauge field which transforms to an ansatz of the same type under $A_i \mapsto g_\Lambda{}^{-1}A_i g_\Lambda + g_\Lambda{}^{-1}\partial_\mu g_\Lambda$, in particular,

$$f_A \mapsto f_A\cos\Lambda - f_B\sin\Lambda, \quad f_B \mapsto f_B\cos\Lambda + f_A \sin\Lambda, \quad f_C \mapsto f_C - r\Lambda'. \tag{45}$$

The requirement of regularity at the origin implies

$$f_A(0) = -1, \quad f_B(0) = 0, \quad f_C(0) = 0, \quad H(0) = \text{const}, \quad K(0) = 0. \tag{46}$$

Also the gauge transformation has to be regular at the origin, $\Lambda(0) = 0$.

In this gauge, inserting the ansatz (41),(44) reduces the static Hamiltonian to the following radial subsystem Hamiltonian:

$$\tilde{H}_0 = 2\pi\left\{\frac{1}{4r}\left[\left(f_A' - \frac{f_B f_C}{r}\right)^2 + \left(f_B' + \frac{f_A f_C}{r}\right)\right] + \frac{\eta^2}{2}\left[r\left(H' - \frac{K f_C}{r}\right) + r\left(K' + \frac{H f_C}{r}\right) + \frac{1}{r}(K f_A - H f_B)^2\right] + \frac{\lambda\eta^4}{4}r(1 - H^2 - K^2)^2\right\}. \tag{47}$$

The Euler-Lagrange equations are

$$\left[r\left(H' - \frac{K f_C}{r}\right)\right]' = f_C\left(K' + \frac{H f_C}{r}\right) - \frac{1}{r}f_B(K f_A - H f_B) - \lambda\eta^2 r H(1 - H^2 - K^2), \tag{48}$$

$$\left[r\left(K' + \frac{H f_C}{r}\right)\right]' = -f_C\left(H' - \frac{K f_C}{r}\right) + \frac{1}{r}f_A(K f_A - H f_B) - \lambda\eta^2 r K(1 - H^2 - K^2), \tag{49}$$

$$\left[\frac{1}{r}\left(f_A' - \frac{f_B f_C}{r}\right)\right]' = \frac{1}{r^2}f_C\left(f_B' + \frac{f_A f_C}{r}\right) + \frac{2\eta^2}{r}K(K f_A - H f_B), \tag{50}$$

$$\left[\frac{1}{r}\left(f_B' + \frac{f_B f_C}{r}\right)\right]' = -\frac{1}{r^2}f_C\left(f_A' - \frac{f_B f_C}{r}\right) - \frac{2\eta^2}{r}H(K f_A - H f_B), \tag{51}$$

and

$$0 = f_A\left(f_B' + \frac{f_B f_C}{r}\right) - f_B\left(f_A' - \frac{f_B f_C}{r}\right) + 2\eta^2 r^2\left[H\left(K' + \frac{H f_C}{r}\right) - K\left(H' - \frac{K f_C}{r}\right)\right]. \tag{52}$$

Note that the ansatz (41),(44) being radially symmetric, Eqs. (48)–(52) are guaranteed consistent with the equations which are obtained by varying the original Hamiltonian (40) and inserting the ansatz (41),(44) afterwards. This is in contrast with the situation in the Weinberg-Salam model where the corresponding ansatz is not spherically symmetric and hence the consistency of the ansatz there must be checked.

As the fifth equation (52) is first order, it is obviously not a dynamical equation, but a constraint on the system which is related to a gauge transformation in the space of the parameter functions $(H, K, f_A, f_B, f_C)$ [26]. This shows that this set of parameter functions has one redundant degree of freedom which is already clear from the fact that we kept the gauge freedom (42),(43),(45) in deriving the reduced Hamiltonian (47). Indeed, the gauge transformation generated by Eq. (52) is equivalent to the transformations (43),(45) derived from the $SO(3)$ gauge transformation $g_\Lambda$ of the original fields $(\Phi, A_i)$ in Eq. (42).

The gauge transformation (43),(45) can now be used to fix $f_C \equiv 0$ without loss of generality in the ansatz (41). This choice corresponds to ''radial gauge'' $\hat{x}_i A_i = 0$. In this gauge, the Hamiltonian (47) simplifies to

$$H_0=2\pi\left\{\frac{1}{4r}(f_A'^2+f_B'^2)+\frac{\eta^2}{2}\left[r(H'^2+K'^2)+\frac{1}{r}(Kf_A-Hf_B)^2\right]+\frac{\lambda\eta^4}{4}r(1-H^2-K^2)^2\right\}. \tag{53}$$

For $f_B\equiv 0$ and $H\equiv 0$ this reduces to the radial subsystem Euclidean Lagrangian of the Abelian Higgs model in two spacetime dimensions (5) supporting the (topologically stable) Abelian Higgs vortex. The embedding of this two-dimensional $SO(2)$-Higgs vortex into the $SO(3)$-Higgs theory as a static object in three spacetime dimensions, $(\Phi,A_i)_{sph}$, yields the sphaleron of the three-dimensional model.

It is interesting to remark that the $SO(3)$ gauge field of this sphaleron tends to one-half times a pure gauge at spatial infinity, $r\to\infty$:

$$(A_i)_{sph}\sim\frac{1}{r}\epsilon_{ik}\hat{x}_k\left(-\frac{i}{2}\sigma_3\right)=\frac{1}{2}g_\pi^{-1}\partial_i g_\pi,$$

$$g_\pi=\exp\left\{\frac{i}{2}\epsilon_{ij}\hat{x}_i\sigma_j\pi\right\}, \tag{54}$$

a property which the $SO(3)$ Higgs sphalerons shares with the instanton of the same model [3]. This is not surprising if we consider that sphalerons and instantons are related objects.

Requiring finite energy

$$\mathcal{E}=\int\mathcal{H}d^2x=\int H_0 dr<\infty \tag{55}$$

fixes the behavior of the remaining parameter functions $(H,K,f_A,f_B)$ at spatial infinity to

$$H(r\to\infty)=\cos q,\qquad f_A(r\to\infty)=-\alpha(q)\cos q,$$

$$K(r\to\infty)=\sin q,\qquad f_B(r\to\infty)=-\alpha(q)\sin q, \tag{56}$$

with $q\in[0,\pi]$.

The relation between $q$ and $\alpha=\alpha(q)$ can be determined from a more careful analysis of the asymptotic behavior of the parameter functions $(H,K,f_A,f_B)$, using the Euler-Lagrange equations of $H_0$ which are

$$\left(\frac{f_A'}{r}\right)'=\frac{2\eta^2}{r}K(Kf_A-Hf_B), \tag{57}$$

$$\left(\frac{f_B'}{r}\right)'=-\frac{2\eta^2}{r}H(Kf_A-Hf_B), \tag{58}$$

$$(rH')'=-\frac{f_B}{r}f_B(Kf_A-Hf_B)-\lambda\eta^2 rH(1-H^2-K^2), \tag{59}$$

$$(rK')'=\frac{f_A}{r}(Kf_A-Hf_B)-\lambda\eta^2 rK(1-H^2-K^2). \tag{60}$$

Of course, these equations agree with Eqs. (48)–(51), setting $f_C=0$. The fifth equation (52) reduces to

$$f_A\left(\frac{f_B'}{r}\right)-f_B\left(\frac{f_A'}{r}\right)+2\eta^2[H(rK')-K(rH')]=0, \tag{61}$$

which turns out to be an integration constant of the other four equations (57)–(60).

The asymptotic analysis then fixes $\alpha(q)=\cos q$, and one obtains the following asymptotic behavior in the region $m_W r\gg 1$:

$$H(r)\sim+\cos q-d_H\cos q(m_W r)^{-1/2}e^{-m_H r}-d_W\cos q\sin q(m_W r)^{-3/2}e^{-m_W r}, \tag{62}$$

$$K(r)\sim+\sin q-d_H\sin q(m_W r)^{-1/2}e^{-m_H r}+d_W\cos^2 q\sin q(m_W r)^{-3/2}e^{-m_W r}, \tag{63}$$

$$f_A(r)\sim-\cos^2 q+D_W\sin q(m_W r)^{1/2}e^{-m_W r}, \tag{64}$$

$$f_B(r)\sim-\cos q\sin q-D_W\cos q(m_W r)^{1/2}e^{-m_W r}, \tag{65}$$

where $D_W$, $d_H$, $d_W$ are constants which can be determined numerically by solving Eqs. (57)–(60). It is easy to check that $m_W r=:\rho$ is the convenient dimensionless radial variable in this model which is also used in all numerical calculations.

Solving Eqs. (57)–(60) in the $m_W r\ll 1$ region yields

$$H(r)\sim c_H+\frac{1}{4}c_H(c_H^2-1)\cdot(m_H r)^2, \tag{66}$$

$$K(r)\sim c_K(m_W r), \tag{67}$$

$$f_A(r)\sim-1+c_A(m_W r)^2, \tag{68}$$

$$f_B(r)\sim c_B(m_W r)^3, \tag{69}$$

with $c_A$, $c_B$, $c_K$ constants which again have to be determined from the numerical integration of the equations of motion, and $c_H=3c_B/c_K$.

Solutions of Eqs. (57)–(60) can only be found for three particular values of $q$, $q\in[0,\pi]$. First, $q=0$ and $q=\pi$ allow the *vacuum configurations* $H\equiv\pm 1$, $K\equiv 0$, $f_A\equiv -1$, $f_B\equiv 0$ with zero energy, resulting in the vacuum fields

$$(\Phi,A_i)_\pm=\left(\pm i\frac{\eta}{\sqrt{2}}\sigma_3,0\right). \tag{70}$$

Second for $q=\pi/2$ the behavior of the parameter functions at infinity, Eqs. (62)–(65), and at the origin, Eqs. (66)–(69), allow one to set $f_B\equiv 0$ and $H\equiv 0$ such that the $SO(2)$ symmetric Abelian Higgs vortex configuration is an element of the subspace of $SO(3)$ Higgs configurations given by the ansatz (41),(44) and the boundary conditions (46),(56) in the radial $f_C\equiv 0$ gauge. This is the sphaleron solution.

To construct a loop from the subspace of $SO(3)$ Higgs configurations discussed above, we have to specify a one parameter subset $(\tilde{\Phi},\tilde{A}_i)_q$ parametrized by the boundary constant $q$. Three points of the loop are already fixed: the initial and final vacua $(\tilde{\Phi},\tilde{A}_i)_{q=0,\pi}=(\Phi,A_i)_{\pm}$ and the sphaleron $(\tilde{\Phi},\tilde{A}_i)_{q=\pi/2}=(\Phi,A_i)_{sph}$ given by the parameter functions $(f_A,K)_{sph}$. We now have to fix the parameter functions $(H,K,f_A,f_B)$ for all remaining values of $q$ such that $q$ becomes the loop parameter.

Instead of $q$, it is convenient to choose the Chern-Simons number as the parameter along the loop. Therefore, we next calculate the Chern-Simons functional for the set of configurations (41),(44) with boundary behavior (62)–(65) at infinity and Eqs. (66)–(69) at the origin, treating $q=q(t)$ as a time-dependent parameter such that a loop parametrized by $q$ starts at an initial vacuum with $q(t=-\infty)=0$ and ends at a final vacuum with $q(t=+\infty)=\pi$, whereas the sphaleron is reached at some time $t_{sph}$, $q(t_{sph})=\pi/2$.

Inserting the ansatz (in radial gauge $f_C\equiv 0$) into the Chern-Simons number functional

$$\mathcal{N}_{CS}(t_0)=\frac{1}{\sqrt{2}\pi\eta}\left\{\int_{\mathbb{R}^2}\Omega_0\Big|_{t=-\infty}^{t=t_0}d^2x+\int_{-\infty}^{t_0}dt\lim_{r\to\infty}\int_{S^1(r)}\Omega_i dS_i\right\}, \quad (71)$$

we obtain

$$\mathcal{N}_{CS}(q)=-\frac{1}{2}\int(Hf_A'+Kf_B')dr+\frac{1}{2}(1-\cos q), \quad (72)$$

with $q=q(t_0)$. This yields $\mathcal{N}_{CS}(q=0)=0$ and $\mathcal{N}_{CS}(q=\pi)=1$; hence the Chern-Simons number increases by unity between the initial and final vacua. For the sphaleron, the surface integral in Eq. (71) yields $\mathcal{N}_{CS}(q=\pi/2)=\frac{1}{2}$.

To complete the loop construction, we minimize the static energy (55) for a fixed value of the Chern-Simons number $\mathcal{N}_{CS}$. This is done by adding the integrand in Eq. (72),

$$N_0=\frac{1}{2}(Hf_A'+Kf_B'), \quad (73)$$

multiplied by a Lagrange multiplier $\xi$, to the reduced Hamiltonian $H_0$ in Eq. (53), and minimizing

$$F_0:=H_0+\xi N_0. \quad (74)$$

The corresponding Euler-Lagrange equations are

$$\left(\frac{f_A'}{r}\right)'=\frac{2\eta^2}{r}K(Kf_A-Hf_B)-\xi H', \quad (75)$$

$$\left(\frac{f_B'}{r}\right)'=-\frac{2\eta^2}{r}H(Kf_A-Hf_B)-\xi K', \quad (76)$$

$$(rH')'=-\frac{f_B}{r}f(Kf_A-Hf_B)-\lambda\eta^2 rH(1-H^2-K^2)+\frac{\xi}{2\eta^2}f_A', \quad (77)$$

$$(rK')'=\frac{f_A}{r}(Kf_A-Hf_B)-\lambda\eta^2 rK(1-H^2-K^2)+\frac{\xi}{2\eta^2}f_B'. \quad (78)$$

The Lagrange multiplier changes the behavior of the solutions at infinity to

$$H(r)\sim+\cos q-d_H\cos q(m_W r)^{-1/2}e^{-\sqrt{m_H^2-(\xi^2/m_W^2)}r}-\xi d_W\sin q(m_W r)^{1/2}e^{-\sqrt{m_W^2-(\xi^2/m_W^2)}r}, \quad (79)$$

$$K(r)\sim+\sin q-d_H\sin q(m_W r)^{-1/2}e^{-\sqrt{m_H^2-(\xi^2/m_W^2)}r}+\xi d_W\cos q(m_W r)^{1/2}e^{-\sqrt{m_W^2-(\xi^2/m_W^2)}r}, \quad (80)$$

$$f_A(r)\sim-\alpha(q)\cos q+\xi D_H\cos q(m_W r)^{1/2}e^{-\sqrt{m_H^2-(\xi^2/m_W^2)}r}+D_W\sin q(m_W r)^{1/2}e^{-\sqrt{m_W^2-(\xi^2/m_W^2)}r}, \quad (81)$$

$$f_B(r)\sim-\alpha(q)\sin q+\xi D_H\sin q(m_W r)^{1/2}e^{-\sqrt{m_H^2-(\xi^2/m_W^2)}r}-D_W\cos q(m_W r)^{1/2}e^{-\sqrt{m_W^2-(\xi^2/m_W^2)}r}, \quad (82)$$

and requiring that exponential decay restricts the range of the Lagrange multiplier to

$$|\xi|<\min\{m_W^2,m_W m_H\}. \quad (83)$$

The asymptotic analysis no longer forces $\alpha(q)=\cos q$, and $\alpha$ along the loop is known only for the vacua and the sphaleron, $\alpha(0)=1$, $\alpha(\pi)=-1$, $\alpha(\pi/2)=0$. The general function $\alpha(q)$ has to be determined from the behavior of the functions $f_A$, $f_B$ at $r\to\infty$.

We first consider the sphaleron and its boundary loop for $\lambda=1$ which happens to be a particularly simple case in the sense that the energy along the loop can be calculated analytically. Choosing

$$f_B\equiv 0, \quad H\equiv\frac{\xi}{2\eta^2} \quad (84)$$

simplifies Eqs. (75)–(78) to

$$\left(\frac{f_A'}{r}\right)'=\frac{2\eta^2}{r}K^2 f_A, \tag{85}$$

$$K'=-\frac{1}{r}Kf_A, \tag{86}$$

$$f_A'=\eta^2 r\left(1-\frac{\xi^2}{4\eta^2}-K^2\right), \tag{87}$$

$$(rK')'=\frac{f_A^2}{r}K-\eta^2 rK\left(1-\frac{\xi^2}{4\eta^2}-K^2\right), \tag{88}$$

while the boundary conditions (46),(56) for the remaining functions $f_A$, $K$ become

$$K(0)=0, \quad K(r\to\infty)=\pm\sqrt{1-\frac{\xi^2}{4\eta^4}},$$
$$f_A(0)=-1, \quad f_A(r\to\infty)=0, \tag{89}$$

as $\cos q=\xi/2\eta^2$ from the boundary condition on $H\equiv\xi/2\eta^2$ at $r\to\infty$.

It is easy to check that Eqs. (86) and (87) are first integrals of Eqs. (85) and (88); hence a consistent solution of the system (85)–(88) can be found by solving the first-order system (86),(87).

Solutions of Eqs. (86),(87) saturate the inequalities

$$\frac{1}{4r}\left[f_A'-\eta^2 r\left(1-\frac{\xi^2}{4\eta^2}-K^2\right)\right]^2\geqslant 0,$$
$$\frac{\eta^2}{2}r\left[K'-\frac{1}{r}Kf_A\right]^2\geqslant 0. \tag{90}$$

The cross terms of the squares yield a lower bound for the Hamiltonian (53) for $\lambda=1$ and $f_B\equiv 0$, $H\equiv\xi/2\eta^2$:

$$H_0|_{\lambda=1,f_B\equiv 0,H\equiv\xi/2\eta^2}=2\pi\left\{\frac{1}{4r}f_A'^2+\frac{\eta^2}{2}\left[rK'^2+\frac{1}{r}(Kf_A)^2\right]+\frac{\eta^4}{4}r\left(1-\frac{\xi^2}{4\eta^4}-K^2\right)^2\right\} \tag{91}$$

$$\geqslant\pi\eta^2\frac{d}{dr}\left\{f_A\left(1-\frac{\xi^2}{4\eta^4}-K^2\right)\right\}. \tag{92}$$

The resulting Hamiltonian (92) is exactly of the same form as the one-dimensional reduced Lagrangian (5) of the Abelian Higgs model, which becomes obvious by replacing the functions $(a,h)$ by $(f_A,K)$ formally, and by replacing 1 by $1-\xi^2/4\eta^4$ in the potential. Solutions to this system [which saturate the bounds (90)] are known numerically.

Exploiting the boundary behavior (89) of these solutions, one can find their energy depending on the Lagrange multiplier $\xi$ from the saturated lower bound (92),

$$\mathcal{E}=\int H_0|_{\lambda=1,f_B\equiv 0,H\equiv\xi/2\eta^2}dr=\pi\eta^2\left(1-\frac{\xi^2}{4\eta^4}\right). \tag{93}$$

Also the integral in the Chern-Simons number (72) depends only on the boundary values for this particular loop, and we obtain

$$\mathcal{N}_{CS}=\frac{1}{2}\left(1-\frac{\xi}{2\eta^2}\right), \tag{94}$$

resulting in an analytic expression for the energy along the boundary loop,

$$\mathcal{E}(\mathcal{N}_{CS})=4\pi\eta^2\mathcal{N}_{CS}(1-\mathcal{N}_{CS}). \tag{95}$$

As a result the slopes of $\mathcal{E}(\mathcal{N}_{CS})$ at the beginning and the end of the loop are

$$\left.\frac{\partial\mathcal{E}(\mathcal{N}_{CS})}{\partial\mathcal{N}_{CS}}\right|_{\mathcal{N}_{CS}=0}=4\pi\eta^2, \quad \left.\frac{\partial\mathcal{E}(\mathcal{N}_{CS})}{\partial\mathcal{N}_{CS}}\right|_{\mathcal{N}_{CS}=1}=-4\pi\eta^2. \tag{96}$$

For coupling constants $\lambda>1$, the full set of equations (75)–(78) has to be solved numerically, using $\rho=\sqrt{2}\eta r$ as rescaled dimensionless radial variable and eliminating $q$ from the boundary conditions at infinity. In practice we replaced the boundary conditions (56) which also involve the unknown function $\alpha(q)$ by

$$H^2+K^2\overset{r\to\infty}{\to}1,$$
$$Kf_A+Hf_B\overset{r\to\infty}{\to}0,$$
$$Hf_A'+Kf_B'\overset{r\to\infty}{\to}0,$$
$$f_A'\overset{r\to\infty}{\to}0, \tag{97}$$

and solved Eqs. (75)–(78) for a given value of $\xi$, identifying $q$ and $\alpha(q)$ from the asymptotic behavior of the numerical solutions afterwards.

The energy $\mathcal{E}$ and Chern-Simons number $\mathcal{N}_{CS}$ can be computed for these solutions. This finally yields a ''static minimal energy loop'' connecting two topologically neighboring vacua through the sphaleron configuration which corresponds to the vortex of the Abelian Higgs model. The energy along the loop which can be parametrized by the Chern-Simons number, $\mathcal{E}(\mathcal{N}_{CS})$, is shown in Fig. 1, for several

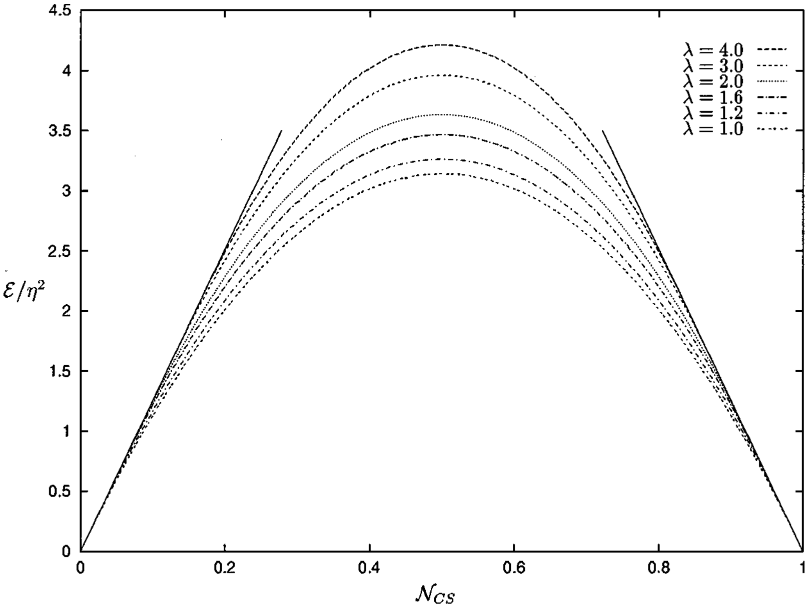


FIG. 1. The energy along the boundary loop, $\mathcal{E}(\mathcal{N}_{CS})$, for $\lambda = 1.0, 1.2, 1.6, 2.0, 3.0, 4.0$. The straight lines show the slopes of the energy curve at the vacua $\mathcal{N}_{CS} = 0,1$.

values of the ratio $m_H^2/m_W^2 = \lambda > 1$. In this parameter region, the Lagrange multiplier is restricted to $|\xi| < \xi_c = 2\eta^2$. $\mathcal{E}(\mathcal{N}_{CS})$ approaches the vacua $\mathcal{E} = 0$ for $\xi \to \mp 2\eta^2$ with slopes

$$\left. \frac{\partial \mathcal{E}(\mathcal{N}_{CS})}{\partial \mathcal{N}_{CS}} \right|_{\mathcal{N}_{CS}=0,1} = \pm 4\pi\eta^2 \tag{98}$$

independently of $\lambda$, because the slope of the energy curve approaching the vacua is given by $\pm 2\pi\xi_c$ [25].

The symmetry $\mathcal{E}(\mathcal{N}_{CS}) = \mathcal{E}(1-\mathcal{N}_{CS})$ is related to the invariance of $H_0$ and $F_0$ in Eqs. (53),(74) under the transformation

$$H \mapsto -H, \quad f_A \mapsto f_A,$$
$$K \mapsto K, \quad f_B \mapsto -f_B, \tag{99}$$

which requires

$$q \mapsto \pi - q, \quad \alpha(\pi - q) = \alpha(q). \tag{100}$$

It is easy to check that this transformation yields

$$\mathcal{N}_{CS} \mapsto 1 - \mathcal{N}_{CS}, \tag{101}$$

while the energy is invariant, yielding the symmetry of the energy along the boundary loop mentioned above.

The boundary loop of the $SO(3)$-Higgs model we constructed is different from the original AKY construction in the Weinberg-Salam model [25]. The Weinberg-Salam sphaleron has a pure gauge connection at infinity. This allowed a boundary loop construction [25] where all gauge fields along the loop tend to a pure gauge; in particular, gauge and Higgs fields at infinity could be constructed as a gauge transformation of the initial vacuum configuration, using the gauge transformation which rotates the parameter fields and allows one to gauge $f_C \equiv 0$. This gauge transformation $g_\Lambda$, Eq. (42), also exists in the $SO(3)$-Higgs boundary loop construction, but it cannot be used to construct the loop configurations at infinity from the initial vacuum. In fact, applying $g_{\Lambda=q}$ to the initial vacuum $(\tilde{\Phi}, \tilde{A}_i)_{q=0} = (\Phi, A_i)_+$ yields the correct behavior of the Higgs field, but not of the gauge field along the loop at spatial infinity,

$$(\tilde{\Phi}, \tilde{A}_i)_{q=0} = \left( i\frac{\eta}{\sqrt{2}}\sigma_3, 0 \right) \overset{g_q}{\mapsto} \left( i\frac{\eta}{\sqrt{2}}(\cos q\,\sigma_3 + \sin q\,\hat{x}_i\sigma_i), \frac{1-\cos q}{r}\epsilon_{ik}\hat{x}_k\left(-\frac{i}{2}\sigma_3\right) - \frac{\sin q}{r}\epsilon_{ij}\hat{x}_j\left(-\frac{i}{2}\hat{x}_k\sigma_k\right)\right), \tag{102}$$

as one can see by comparing with the Higgs and gauge fields along the loop at spatial infinity, i.e.,

$$(\tilde{\Phi}, \tilde{A}_i)_q \overset{r\to\infty}{\sim} \left( i\frac{\eta}{\sqrt{2}}(\cos q\,\sigma_3 + \sin q\,\hat{x}_i\sigma_i), \times\frac{1-\alpha(q)\cos q}{r}\epsilon_{ik}\hat{x}_k\left(-\frac{i}{2}\sigma_3\right) - \frac{\alpha(q)\sin q}{r}\epsilon_{ij}\hat{x}_j\left(-\frac{i}{2}\hat{x}_k\sigma_k\right)\right). \tag{103}$$

This is due to the fact that the sphaleron, which has to be a loop configuration, is one-half times a pure gauge at infinity in accordance with the corresponding behavior of the instanton.

We expect that this is a general feature of models which support both instantons and sphalerons, i.e., the sphaleron behaves like the instanton at spatial infinity, a fact which must be reflected in the boundary loop construction.

### B. General vorticity sphalerons and geometrical loop construction

In general, the Abelian Higgs vortex has vorticity or winding number $N\in\mathbb{N}$. So far, we considered only $N=1$ for the $SO(3)$ sphaleron, but the generalization to sphalerons of arbitrary integer vorticity can easily be achieved. We use this generalization in our presentation of the second geometrical loop construction.

The starting point of the geometrical loop construction is the sphaleron ansatz which we generalize to vorticity $N$, replacing the spatial unit vector $\hat{x}$ by $\hat{n}=(\cos N\phi,\sin N\phi)$ in the ansatz,

$$\Phi=i\frac{\eta}{\sqrt{2}}h(r)\hat{n}_i\sigma_i, \quad A_i=\frac{N}{r}f(r)\epsilon_{ik}\hat{x}_k\left(-\frac{i}{2}\sigma_3\right), \tag{104}$$

where we use $h(r)$ and $f(r)$ as parameter functions [4]. Inserting the ansatz into the Hamiltonian (40) yields the radial subsystem Hamiltonian

$$H_{sph}=2\pi\left\{\frac{1}{4}\frac{N^2}{r}f'^2+\frac{\eta^2}{2}\left[rh'^2+\frac{N^2}{r}h^2(1-f)^2\right]+\frac{\lambda\eta^4}{4}r(1-h^2)^2\right\}. \tag{105}$$

Finite energy and regularity at the origin require

$$f(r\to\infty)=1, \quad h(r\to\infty)=1, \quad f(0)=0, \quad h(0)=0, \tag{106}$$

and the solutions are equivalent to the vorticity $N$ Abelian Higgs vortices as expected.

The geometrical loop construction starts from requiring finite energy; hence the potential term in the Hamiltonian (40) forces $|\Phi|=\eta\Rightarrow\Phi\in S^2_{Higgs}$ at spatial infinity $(\mathbb{R}^2)^\infty=S^1_{space}$, described by the angular coordinate $\phi$. Therefore we consider the Higgs field for $r\to\infty$ and add a single loop parameter $\tau\in S^1_{loop}$ to construct a nontrivial mapping,

$$\Phi^\infty:S^1_{space}\times S^1_{loop}\to S^2_{Higgs}. \tag{107}$$

A simple choice for this mapping with vorticity $N$ is

$$\Phi^\infty=i\frac{\eta}{\sqrt{2}}\vec{p}\cdot\vec{\sigma}, \quad \vec{p}=\begin{pmatrix}\sin\tau\cos N\phi\\ \sin^2\tau\sin N\phi+\cos^2\tau\\ \sin\tau\cos\tau(\sin N\phi-1)\end{pmatrix}. \tag{108}$$

This fixes also the gauge field at infinity since the covariant derivative in the Hamiltonian (40) has to vanish, requiring

$$A_i^\infty:=A_i(r\to\infty)=-\frac{1}{2\eta^2}[\Phi^\infty,\partial_i\Phi^\infty]. \tag{109}$$

According to the results of Sec. III A, the gauge field at infinity along the loop is not a pure gauge; in particular, it tends to half a pure gauge for $r\to\infty$ for the sphaleron, $\tau=\pi/2$.

For $\tau=0$ and $\tau=\pi$, the loop has to start and end in the vacuum which for convenience [4] we choose to be

$$(\Phi,A_i)_{vac}=\left(i\frac{\eta}{\sqrt{2}}\sigma_2,0\right), \tag{110}$$

in contrast to the choice of vacua in Sec. III A. This yields the geometrical NCL ansatz

$$(\overline{\Phi},\overline{A}_i)_\tau=\left(i\frac{\eta}{\sqrt{2}}\{[1-h(r)]\vec{t}+h(r)\vec{p}\},\; -\frac{1}{2\eta^2}f(r)A_i^\infty\right) \quad \text{with} \quad \vec{t}=\begin{pmatrix}0\\ \cos^2\tau\\ -\sin\tau\cos\tau\end{pmatrix}. \tag{111}$$

The boundary conditions (106) then ensure

$$(\overline{\Phi},\overline{A}_i)_\tau\overset{r\to\infty}{\longrightarrow}(\Phi^\infty,A_i^\infty), \tag{112}$$

and $\vec{t}$ is chosen such that

$$(\overline{\Phi},\overline{A}_i)_{\tau=0}=(\overline{\Phi},\overline{A}_i)_{\tau=\pi}=(\overline{\Phi},\overline{A}_i)_{vac}. \tag{113}$$

One can also check that $\Phi^2$ is a radial function of $r$ only which is necessary for the consistency of the ansatz.

Inserting the loop ansatz (111) into the static Hamiltonian (40) yields

$$H_\tau=2\pi\sin^3\tau\left\{\frac{1}{4}\frac{N^2}{r}f'^2+\frac{\eta^2}{2}\left[rh'^2+\frac{N^2}{r}\times[h-f(h\sin^2\tau+\cos^2\tau)]^2\right]+\frac{\lambda\eta^4}{4}\sin^2\tau r(1-h^2)^2\right\}. \tag{114}$$

For $\tau=\pi/2$, this geometrical loop ansatz reduces to the sphaleron ansatz (104), and of course $H_{\tau=\pi/2}=H_{sph}$, whereas $H_{\tau=0}=0=H_{\tau=\pi}$ for the vacua at the beginning and the end of the loop.

$H_\tau$ can be minimized numerically for any value $\tau\in[0,\pi]$, yielding again a ''static minimal energy path'' connecting two neighboring vacua through the sphaleron, the energy along the loop being $\mathcal{E}(\tau)=\int H_\tau dr$.

One can also calculate the Chern-Simons number along the NCL, treating $\tau=\tau(t)$ as time dependent with $\tau(-\infty)$

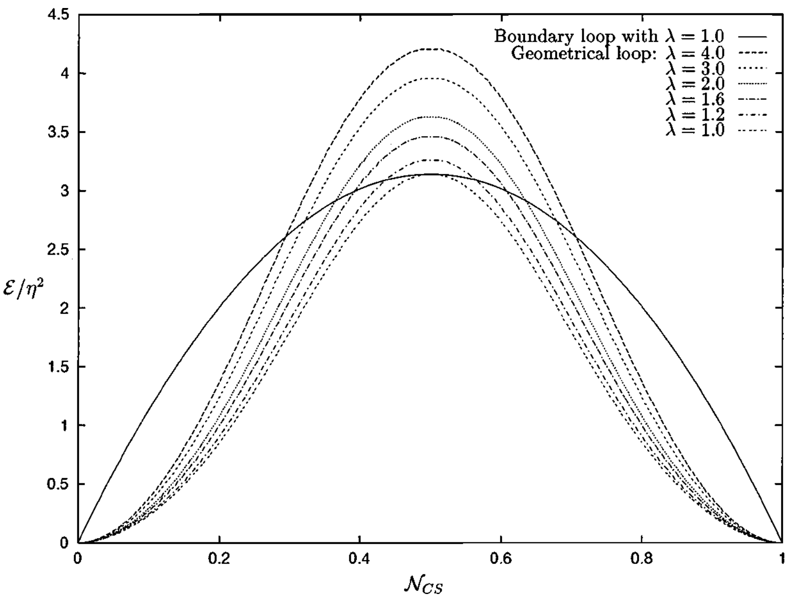


FIG. 2. The energy along the geometrical loop, $\mathcal{E}(\mathcal{N}_{CS})$, with vorticity $N=1$ and $\lambda=1.0,1.2,1.6,2.0,3.0,4.0$, and the energy along the boundary loop for $N=1$, $\lambda=1.0$.

$=0$, $\tau(\infty)=\pi$. For the Manton loop ansatz (111), both the volume and the surface integrals in Eq. (71) contribute, resulting in

$$\mathcal{N}_{CS}(\tau)=\frac{N}{2}\cos\tau\sin^2\tau\int f'(1-h)dr+\frac{N}{2}(1-\cos\tau). \tag{115}$$

For the sphaleron $\tau=\pi/2$, only the surface integral contributes to $\mathcal{N}_{CS}=N/2$, whereas the loop as a whole has $\mathcal{N}_{CS}=N$. The vorticity $N$ sphaleron therefore is the saddle point of a loop connecting to vacua with Chern-Simons number difference $N$.

The "static minimal energy path" of the $SO(3)$ gauge Higgs sphaleron in the geometrical construction, $\mathcal{E}(\mathcal{N}_{CS})$, is shown in Fig. 2 with vorticity $N=1$ and several values of $\lambda$. Comparing the $\lambda=1$ geometrical loop with the corresponding boundary loop, we find that the energy along the geometrical loop is lower than that along the boundary loop except at the sphaleron and the vacua where they are equal. This result holds also for $\lambda>1$.

Figure 3 shows the geometrical loop for several values of the vorticity.

### C. Magnetic and electric properties of Georgi-Glashow sphalerons

Since the effective equations governing the sphaleron solution of the three-dimensional Georgi-Glashow model are those of the Abelian Higgs vortex with vorticity $N$ (hence also magnetic charge), we might expect that the sphaleron itself may carry magnetic charge.

The electromagnetic field strength of the three-dimensional Georgi-Glashow model is the well-known 't Hooft electromagnetic tensor [15,18]

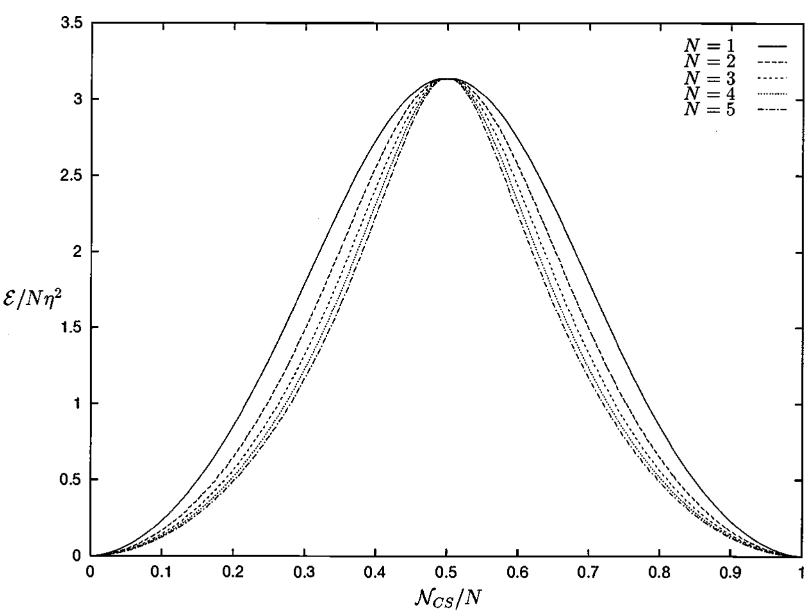


FIG. 3. The energy along the geometrical loop, $\mathcal{E}(\mathcal{N}_{CS}/N)/N$, with $\lambda=1.0$ and vorticity $N=1,2,3,4,5$.

$$\mathcal{F}_{\mu\nu} := -\frac{\mathrm{tr}[\Phi F_{\mu\nu}]}{\sqrt{-\mathrm{tr}[\Phi^2]}} = \frac{\vec{\phi}\cdot\vec{F}_{\mu\nu}}{|\vec{\phi}|}. \tag{116}$$

Inserting the sphaleron configuration (104) into Eq. (116) yields $\mathcal{F}_{\mu\nu}\equiv 0$; therefore the sphaleron itself carries neither magnetic nor electric charge. The situation is different if we consider the complete NCL as the configuration dependent on the time parameter. For example, along the boundary loop presented in Sec. III A with $q=q(t)$ we have nonvanishing electric and magnetic fields,

$$B = \mathcal{F}_{12} = -\frac{1}{r}\frac{Hf_A' + Kf_B'}{\sqrt{H^2+K^2}}, \tag{117}$$

$$E_i = \mathcal{F}_{0i} = -\epsilon_{ij}\frac{\hat{x}_j}{r}\frac{H\dot{f}_A + K\dot{f}_B}{\sqrt{H^2+K^2}}. \tag{118}$$

It follows, by symmetry, that the electric charge of the NCL,

$$Q_e := \lim_{r\to\infty}\int_{S(r)} \vec{E}\cdot d\vec{S}, \tag{119}$$

is always zero, whereas the NCL configurations may acquire magnetic charge

$$Q_m := \int B d^2x \tag{120}$$

at some values of the time parameter. Clearly, the magnetic charge of the sphaleron and the initial and final vacua vanish according to the values that the functions $f_A, f_B, K$, and $H$ take for these configurations. The same conclusion is arrived at also by inspecting the magnetic flux of the vorticity $N$ configuration calculated in terms of the geometric loop parameter $\tau$:

$$Q_m = \sin^2\tau\cos\tau N\int_0^\infty (1-h)f'\,dr, \tag{121}$$

which vanishes at the sphaleron, $\tau=\pi/2$, and the vacua $\tau=0$ and $\tau=\pi$.

One might think at this point that there is no reason why there should be no nonvanishing electric flux if the temporal gauge condition $A_0=0$ were relaxed in the spirit of the Julia-Zee dyon [27]. In the $A_0\neq 0$ one can solve the full Euler-Lagrange equations in Minkowski space, of

$$\mathcal{L}_M = \mathrm{tr}\left[-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}D_\mu\Phi D^\mu\Phi - \lambda\left(\phi^2 + \frac{\eta^2}{2}\right)^2\right], \tag{122}$$

in the static limit using the radially symmetric restriction of the fields according to the ansatz

$$\Phi = i\frac{\eta}{\sqrt{2}}h(r)\hat{x}_i\sigma_i,$$

$$A_0 = i\frac{\eta}{\sqrt{2}}g(r)\hat{x}_i\sigma_i, \quad A_i = \frac{f(r)}{r}\epsilon_{ij}\hat{x}_j\left(-\frac{i}{2}\sigma_3\right). \tag{123}$$

To find solutions that lead to nonvanishing electric flux $Q_e$, the solution must have the following asymptotic behavior:

$$g(r) \sim d_g + \sqrt{2}\pi\eta Q_e \ln r, \quad m_W r \gg 1, \tag{124}$$

where $d_g$ is a constant. The asymptotic analysis of the equations, not exhibited here, indeed confirms the behavior (124). These equations have been integrated in [18] numerically.

However, this static electrically charged solution cannot be called a sphaleron since the logarithmic behavior of $g(r)$ (which is a consequence of $\ln r$ being the fundamental solution of the Laplace operator in two dimensions) destroys the integrability of the static Hamiltonian; i.e., this electrically charged classical configuration does not have finite energy, rendering it useless as a pseudoparticle in quantum field theory. Technically, this is related to the well-known fact that the energy integral of point charges in two-dimensional electrodynamics is divergent.

## IV. SUMMARY AND DISCUSSION

We have analyzed the first two in the hierarchy of $SO(d)$-gauged $d$-dimensional Higgs models where the Higgs fields are in the $d$-dimensional vector representation of $SO(d)$. In $d=2$ and 3 these are the familiar Abelian Higgs model and the Georgi-Glashow model, respectively. The reason for having chosen these models is that they both support topologically stable finite action solutions which we interpret as the instantons, and they are expected to support sphaleron solutions in the static limit, which we find indeed to be the case. Thus we have analyzed the first two Higgs models in this hierarchy, which support both instantons and sphalerons. This is the main criterion of the work, and our motivations for it are explained in the Introduction. Our analysis points the way to tackling the $d=4$ case [12] which is of some definite physical relevance in as far as it promises a Coulomb instanton gas [13], but which is considerably more complex.

Since the instantons of these two models are well-known solutions, namely, the Nielsen-Oleson vortices [14] and the ’t Hooft-Polyakov monopole [15], the analysis of this paper is restricted to the study of the sphalerons of these models.

Indeed, the sphalerons of the Abelian Higgs model have been studied extensively in the literature [16,17,20], but we give our own version here. The reason for this is first so that the construction of the sphalerons of both models should proceed in the same lines, especially since one of criteria is to map the way to carry this analysis to the $d=4$ case. Second, our construction of the sphaleron of the Abelian Higgs model follows exactly the same procedure as those used in the corresponding work for the Weinberg-Salam model, namely, that of constructing a NCL carried out by Manton [4], and also that of constructing the path of finite energy configurations carried out by Akiba *et al.* [25]. This contrasts with the presentation in the literature [16,17,20] where the procedure is quite different from the case of the Weinberg-Salam model [4,5,25], in particular imposing periodic boundary conditions [16] unnecessarily. This part of our work is presented in Sec. II, where we have given both the construction of the sphaleron, namely, that of Klinkhamer and Manton [4,5] and that of Akiba *et al.* [25]. In addition, we have made a contrast of our procedures with those existing in the literature. Our procedure here, making it possible to formulate this problem in complete parallel to the Weinberg-Salam case, is made possible by our use of the systematically derived [19] Chern-Pontryagin density used in calculating the Chern-Simons number.

In the larger part of this work we have presented the construction of the sphaleron in the three-dimensional Georgi-Glashow model. This is the work in Sec. III. Again we have presented the construction of the sphaleron in both the Klinkhamer-Manton [4,5] procedure and the one of Akiba *et al.* [25]. It is very interesting that in the former case [4,5], we find that the sphaleron is a solution to the effective Abelian Higgs model in two spatial dimensions, embedded in the $SO(3)$ model. In that case we have constructed a family of sphalerons characterized by a vortex number *N*, availing ourselves of the fact that radially symmetric fields in two dimensions have integer vorticity. We have also inquired whether this sphaleron has magnetic flux related to its vorticity *N* and have found that the sphaleron itself, as well as the vacua it falls between, have zero magnetic flux, but that intermediate field configurations do have nonvanishing magnetic flux. We have also verified that by relaxing the temporal gauge, one can find solutions to the static field equations leading to a nonvanishing electric flux [18], but in that case the energy diverges logarithmically and hence there is no sphaleron solution that can be gainfully employed in semiclassical field theory. In addition to this, we have pursued the construction in the procedure of Akiba *et al.* [25] and have constructed the finite energy path interpolating between the two vacua with the sphaleron at the top of this path with Chern-Simons number equal to $\frac{1}{2}$. An interesting circumstance here is that when the coupling constant of the Higgs self-interaction potential in the Georgi-Glashow model takes the critical value for which the (static) embedded Abelian Higgs model can saturate the corresponding Bogomol'nyi bound, the finite energy path can be constructed analytically without recourse to numerical computations.

Our analysis of these two models has clarified the similarities and differences between the sphalerons of the Weinberg-Salam model and those of the models in our hierarchy of $SO(d)$-Higgs models. In this respect, the $d=3$ case is the most enlightening. We have learned that the structure of the geometric loop [4,5] behaves very much like that of the Weinberg-Salam model, and that the boundary loop [25] also behaves similarly in spite of the fact that the actual boundary conditions are quite different in the two cases. In the process, we have given a unified treatment for the construction of energy loops interpolating between vacua for all models, bringing the treatment of the Abelian Higgs case into line with the others. This we have done employing both types of energy loops, geometrical [4] and boundary [25], from which we have learned that the geometric loop is lower than the boundary loop everywhere except at the sphaleron and the vacua where they are equal. Presumably this is the case also for the Weinberg-Salam model.

As a final remark we emphasize the similarity between the boundary loop construction for the Winberg-Salam and the Georgi-Glashow models—in both of them the functions $f_B$ and $H$ are excited on the boundary loop. In addition both theories tend to sigma models in the limit of the Higgs boson masses becoming infinit. With these two similarities in place, it seems that there may well be bisphalerons [28,29] in the static three-dimensional Georgi-Glashow model.

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